

POINCARÉ ANOMALY IN PLANAR FIELD THEORY

Subir = Ghosh¹

P. A. M. U., Indian Statistical Institute,
203 B. T. Road, Calcutta 700035, India.

Abstract:

We show the presence of Poincaré anomaly in Maxwell-Chern-Simons theory = with an explicit mass term, in 2+1-dimensions.

¹email:sghosh@isical.ac.in;subir@boson.bose.res.in

The ubiquitous use of 2+1-dimensional field theories in condensed matter systems, where the dynamics normal to a plane is severely restricted, has enlarged the scope of lower dimensional physics from being just a toy model of the 3+1-dimensional world. The topologically massive Maxwell-Chern-Simons (MCS) gauge theory was first thoroughly analysed by Deser, Jackiw and Templeton (DJT) in their seminal work [1], where the subtle interplay between Poincare invariance and an unambiguous determination of the spin of the excitations in a vector theory was revealed. It was shown that correct space-time transformation of the gauge invariant observables, such as electric and magnetic fields, were induced by Poincare generators which obey an anomalous algebra among themselves. However, a phase re-definition of the creation and annihilation operators removed the commutator anomaly and yielded the spin contribution in a single stroke. In the present Letter, we consider the MCS model with an explicit mass term, *i.e.* the MCS-Proca (MCSP) model. We show that in the presence of two mass scales, the topological one (μ), generated by the Chern-Simons term, and the explicit (gauge symmetry breaking one) (m), the anomaly in the Poincare transformations in the electromagnetic fields can not be removed, even though the equations of motion are manifestly Lorentz covariant. This is our main result.

The MCSP model was studied previously in [2]. It also appears naturally in the large fermion mass limit of the bosonization of gauged massive Thirring model [3]. In [2], it was argued that the self-dual factorisation of the equation of motion leads to two self and anti-self dual excitations of different masses, thereby accounting for the parity violation induced by the topological term. Recent results [4] indicate, in the path integral formalism, that the above naive conclusions are invalid. As shown in [4], the fact that MCSP model is a result of a fusion between self and anti-self dual models explains the self dual factorisation of the equation of motion. But in the process, the self and anti-self dual property of the MCSP model is no longer manifest. This controversy demands an indepth analysis of the model.

The MCSP model, with the metric being $g_{\mu\nu} = 3D \text{diag}(+ - -)$, $\epsilon_{12} = 3D1$, is

$$\mathcal{L}_{MCSP} = 3D - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}A^{\mu\nu}A^\lambda + \frac{m^2}{2}A_\mu A^\mu, \quad A_{\mu\nu} = 3D\partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

Taking $m^2 = 3D0$ reproduces the MCS theory, which being a gauge theory is amenable to gauge fixing conditions. This simplifies the model considerably and makes the field content transparent. We also try to implement similar parametrizations as in [1] and hence convert the above gauge non-invariant theory to a gauge invariant one by the Stuckelberg prescription,

$$\mathcal{L}_{St} = 3D - \frac{1}{4}A_{\mu\nu}A^{\mu\nu} + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}A^{\mu\nu}A^\lambda + \frac{m^2}{2}(A_\mu + \partial_\mu\theta)(A^\mu + \partial^\mu\theta), \quad (2)$$

where θ is the Stuckelberg field. We define the conjugate momenta [1] and the Poisson bracket algebra as,

$$\begin{aligned} \frac{\partial \mathcal{L}_{St}}{\partial \dot{A}^i} \equiv \Pi^i &= 3D - \dot{A}_i + \partial_i A_0 - \frac{\mu}{2}\epsilon_{ij}A_j; \quad \frac{\partial \mathcal{L}_{St}}{\partial \dot{A}^0} \equiv \Pi^0 = 3Dm^2\theta; \quad \frac{\partial \mathcal{L}_{St}}{\partial \dot{\theta}} \equiv \Pi_\theta = 3Dm^2\dot{\theta}, \\ \{A_\mu(x), \Pi_\nu(y)\} &= 3D - g_{\mu\nu}\delta(x-y), \quad \{\theta(x), \Pi_\theta(y)\} = 3D\delta(x-y). \end{aligned} \quad (3)$$

The Hamiltonian is

$$\begin{aligned}
\mathcal{H}_{St} &= 3D\Pi^\mu \dot{A}_\mu + \Pi_\theta \dot{\theta} - \mathcal{L}_{St} \\
&= 3D\frac{1}{2}\Pi_i^2 + \frac{1}{4}A_{ij}A_{ij} + \left(\frac{m^2}{2} + \frac{\mu^2}{8}\right)A_iA_i = 20 - \frac{\mu}{2}\epsilon_{ij}\Pi_iA_j \\
&+ \frac{1}{2m^2}\Pi_\theta^2 + \frac{m^2}{2}\partial_i\theta\partial_i\theta + m^2(\partial =_i A_i)\theta - A_0(\partial_i\Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial =_i A_j + \frac{m^2}{2}A_0), \tag{4}
\end{aligned}$$

where a total derivative term has been dropped. The two involuting first = class constraints, (in the Dirac sense of classification), are

$$\chi_1 \equiv \Pi_0 - m^2\theta, \quad \chi_2 \equiv \partial_i\Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial_iA_j + m^2A_0 + \Pi_\theta. \tag{5}$$

The unitary gauge, $\phi_1 \equiv \Pi_\theta$; $\phi_2 \equiv \theta$, establishes gauge equivalence between the embedded model and the = original MCSP model. This ensures that in the gauge invariant sector, = results obtained in any convenient gauge will be true for the MCSP = theory. We invoke the rotationally symmetric Coulomb gauge [1]

$$\psi_1 \equiv A_0; \quad \psi_2 \equiv \partial_iA_i. \tag{6}$$

The (χ_i, ψ_j) system of four constraints are now second = class, meaning that the constraint algebra matrix is invertible. The = Dirac brackets, defined in the conventional way are given below,

$$\begin{aligned}
\{A_i(x), \Pi_j(y)\}^* &= 3D(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta(x-y); \quad \{\Pi_i(x), \Pi_j(y)\}^* = 3D - \frac{\mu}{2}\epsilon_{ij}\delta(x-y) \\
\{\Pi_i(x), \theta(y)\}^* &= 3D\frac{\partial_i}{\nabla^2}\delta(x-y); \quad \{\Pi_i(x), \Pi_0(y)\}^* = 3D - m^2\frac{\partial_i}{\nabla^2}\delta(x-y). \tag{7}
\end{aligned}$$

The remaining brackets are same as the Poisson brackets. The reduced = Hamiltonian in Coulomb gauge is

$$\mathcal{H}_S = 3D\frac{1}{2}\Pi_i^2 + \frac{1}{2}\partial_iA_j\partial =_i A_j + \left(\frac{m^2}{2} + \frac{\mu^2}{8}\right)A_iA_i = 20 - \frac{\mu}{2}\epsilon_{ij}\Pi_iA_j + \frac{1}{2m^2}\Pi_\theta^2 + \frac{m^2}{2}\partial_i\theta\partial_i\theta. \tag{8}$$

Although somewhat tedious, it is straightforward to verify that the = following combinations, $\phi = 3D((\epsilon_{ij}\partial_iA_j), (\epsilon_{ij}\partial_i\Pi =_j), \Pi_\theta, \theta)$ obey the higher derivative equation

$$(\square + M_1^2)(\square + M_2^2)\phi = 3D0; \quad M_1^2(M_2^2) = 3D\frac{1}{2}[2m^2 + \mu^2 \pm \mu\sqrt{\mu^2 + 4m^2}]. \tag{9}$$

The spectra agrees with [2]. Note that for $\mu^2 = 3D0$, the roots collapse to $M_1^2 = 3DM_2^2 = 3Dm^2$, which is = just the Maxwell-Proca model, whereas for $m^2 = 3D0$ the roots are $M_1^2 = 3D\mu^2$, $M_2^2 = 3D0$, indicating the = presence of only the topologically massive mode, since the Stuckelberg = field θ is no longer present.

Prior to fixing the ψ_2 gauge, the gauge invariant sector is identified as,

$$E_i = 3D - \Pi_i + \frac{\mu}{2}\epsilon_{ij}A_j; \quad B = 3D - \epsilon =_{ij} \partial_i A_j; \quad \Pi_\theta; \quad A_i + \partial_i \theta, \quad (10)$$

where E_i and B are the conventional electric and magnetic field. In the reduced space, the Hamiltonian and spatial translation generators are gauge invariant,

$$\mathcal{H}_{St} = 3D \frac{1}{2} (E_i^2 + B^2 + \frac{\Pi_\theta^2}{m^2} + m^2 (A_i + \partial_i \theta)^2),$$

$$\mathcal{P}_{St}^i = 3D - \epsilon_{ij} E_j B - \Pi_\theta (A_i + \partial_i \theta). \quad (11)$$

Defining the boost transformation as $M^{i0} = 3D - t \int d^2x \mathcal{P}_{St}^i(x) + \int d^2x x^i \mathcal{H}_{St}(x)$, the Dirac brackets with the gauge invariant variables are easily computed. They will contain non-canonical pieces in order to be consistent with the constraints. However, changing to a new set of variables by the following canonical transformations,

$$Q_1(Q_2) = 3D \frac{1}{\sqrt{-2\nabla^2}} [\epsilon_{ij} \partial_i A_j \pm \frac{1}{m} \Pi_\theta]; \quad P_1(P_2) = 3D [\frac{1}{\sqrt{-2\nabla^2}} \epsilon_{ij} \partial_i \Pi_j \mp \frac{m}{2} \sqrt{-2\nabla^2} \theta], \quad (12)$$

we can convert our system to a nearly decoupled one. Passing on to the quantum theory, the redefined variables satisfy the canonical algebra,

$$i\{P_i, Q_j\} = 3D \delta_{ij} \delta(x - y); \quad \{Q_i, Q_j\} = 3D \{P_i, P_j\} = 3D 0. \quad (13)$$

The electric and magnetic fields and the translation generators are rewritten as,

$$B = 3D - \frac{\sqrt{-2\nabla^2}}{2} (Q_1 + Q_2); \quad E_i = 3D - \frac{1}{\sqrt{-2\nabla^2}} [\epsilon_{ij} \partial_j (P_1 + P_2) + (\mu + m) \partial_i Q_1 + (\mu - m) \partial_i Q_2], \quad (14)$$

$$H_{St} = 3D \int d^2x [\frac{1}{2} (P_1^2 + \partial_i Q_1 \partial_i Q_1 + M_1^2 Q_1^2) + \frac{1}{2} (P_2^2 + \partial_i Q_2 \partial_i Q_2 + M_2^2 Q_2^2) + \frac{\mu^2}{2} Q_1 Q_2]$$

$$P_{St}^i = 3D \int d^2x [P_1 \partial^i Q_1 + P_2 \partial^i Q_2] \quad (15)$$

In order to drive home the peculiarities of MCSP theory, let us briefly consider the special cases, $m^2 = 3D 0$ or $\mu^2 = 3D 0$. In the former limit, giving the MCS theory, as we noted before, θ field is absent, which makes the (Q_1, P_1) pair identical to the (Q_2, P_2) pair, leading to the following relations, with $i[p(x), q(y)] = 3D \delta(x - y)$,

$$B = 3D \sqrt{-\nabla^2} q, \quad E_1 = 3D \frac{1}{\sqrt{-\nabla^2}} (\epsilon_{ij} \partial_j p + \mu \partial_i q),$$

$$H = 3D \int d^2x \frac{1}{2} (p^2 + \partial_i q \partial_i q + \mu^2 q^2), \quad P^i = 3D \int d^2x (p \partial^i q). \quad (16)$$

This set of relations is identical to those in [1] and hence the results obtained by DJT will follow trivially.

The latter case, $\mu^2 = 3D0$, refers to the Proca model, where $= M_1^2 = 3DM_2^2 = 3Dm^2$, and we get,

$$B = 3D - \frac{\sqrt{-2\nabla^2}}{2}(Q_1 + Q_2); \quad E_i = 3D - \frac{1}{\sqrt{-2\nabla^2}}[\epsilon_{ij}\partial_{=j}(P_1 + P_2) + m(\partial_i Q_1 - \partial_i Q_2)],$$

$$H = 3D \int d^2x \left[\frac{1}{2}(P_1^2 + \partial_i Q_1 \partial_{=i} Q_1 + m_1^2 Q_1^2) + \frac{1}{2}(P_2^2 + \partial_i Q_2 \partial_{=i} Q_2 + m_2^2 Q_2^2) \right],$$

$$P^i = 3D \int d^2x [P_1 \partial^i Q_1 + P_2 \partial^i Q_2]. \quad (17)$$

Following the prescription of DJT given in [1], the boost = generator M^{i0} should be reinforced by the additional terms,

$$m\epsilon_{ij} \int d^2x \left(\frac{P_1 \partial_j Q_1}{-\nabla^2} - \frac{P_2 \partial_j Q_2}{-\nabla^2} \right),$$

such that the electromagnetic fields transform correctly. This addition, = however, generates a zero momentum anomaly in the boost algebra,

$$i[M^{i0}, M^{j0}] = 3D\epsilon^{ij}(M - \Delta), \quad \Delta = 3D \frac{m^3}{4\pi} \left\{ \left(\int Q_1 \right)^2 - \left(\int Q_2 \right)^2 \right\} + \frac{m}{4\pi} \left\{ \left(\int P_1 \right)^2 - \left(\int = P_2 \right)^2 \right\}, \quad (18)$$

where M is the rotation generator=20

$$M = 3D - \int d^2x (P_1 \epsilon^{ij} x^i \partial_{=j} Q_1 + P_2 \epsilon^{ij} x^i \partial_j Q_2)$$

. Making the mode expansions,

$$Q_1(x)(Q_2(x)) = 3D \int \frac{d^2k}{2\pi\sqrt{2\omega_{=}(k)}} [e^{-ikx} a(k)(b(k)) + e^{ikx} a^+(k)(b^+(k))], \quad (19)$$

and effecting the phase redefinitions,

$$a \rightarrow e^{i\frac{m}{|m|}\theta} a, \quad b \rightarrow e^{-i\frac{m}{|m|}\theta} b, \quad (20)$$

where $\theta = 3D \tan^{-1} k_2/k_1$, one recovers the full angular = momentum as

$$M = 3D \int d^2k (a^+(k) \frac{1}{i} \frac{\partial}{\partial \theta} a(k) + b^+(k) \frac{1}{i} \frac{\partial}{\partial \theta} b(k)) + \frac{m}{|m|} \int d^2k (a^+(k)a(k) - b^+(k)b(k)), \quad (21)$$

where the second term is the spin.

Now comes the intriguing part, *i.e.* what happens when both μ and m are nonzero. First of all, for simplicity, let us neglect $O(\mu^2)$ terms, which makes \mathcal{H}_{St} a decoupled sum of "1" and "2" variables. But even then, similar extensions in M^{i0} , as done in the previous cases, will not have the desired effect since the parameters present in \mathcal{H}_{St} , $M_1^2(M_2^2) |_{\mu^2=3D0} = 3Dm^2 \pm m\mu$ are different from the parameters appearing in = the electric field, $(\mu \pm m)^2 |_{\mu^2=3D0}$. Obviously, if we = keep the $O(\mu^2)$ terms as well, the situation will worsen since = \mathcal{H}_{St} is no longer decoupled. This constitutes the main result of this Letter.

In an earlier work [5], it was argued that the anomaly in [1] appeared only because of the mapping of the system in terms of a scalar variable. However, it has been demonstrated in [1] how to overcome this problem, leading to the correct spin value of the excitation in the process. As we have shown, this scheme is untenable in the MCSP model.

To conclude, We have shown that in the Maxwell-Chern-Simons-Proca model, where two mass scales, topological and non-topological or explicit, are present simultaneously, the electromagnetic field transforms anomalously under Poincare transformations. The conventional way [1] of redefining the phases of the creation and annihilation operators of the basic fields to remove the anomaly is inadequate in the present case. A deeper understanding of this pathological behaviour is necessary. However, in applications of condensed matter physics, where Poincare or Lorentz invariance is generally not a big issue, these models can still play an important role.

References

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